Math 579 Fall 2013 Exam 4 Solutions

1. Prove that $\binom{2n}{n}$ is composite for all integers $n \geq 2$.

   Combinatorial proof: We pair each subset of size n from $[2n]$ with its complement, which is a different subset of size n. Hence such subsets come in pairs, and there are therefore an even number of them.

2. Calculate $\sum_{0 \leq k \leq 19} \binom{3-k}{4}$.

   We first use upper negation ($4 \in \mathbb{Z}$) to get $\binom{3-k}{4} = (-1)^4 \binom{4-(3-k)-1}{4} = \binom{k}{4}$. We now use summation on the upper index ($4, 19 \in \mathbb{N}_0$) to get $\sum_{0 \leq k \leq 19} \binom{k}{4} = \binom{20}{5} = 15,504$.

3. For $n \in \mathbb{N}$, calculate $\sum_k k^2 \binom{n}{k}^2$.

   By absorption ($k \in \mathbb{Z}$), we have $k\binom{n}{k} = n\binom{n-1}{k-1}$ so our sum becomes $n^2 \sum_k \binom{n-1}{k-1}^2$. We could either apply a variant of Vandermonde that we proved in class, or use symmetry ($n \in \mathbb{N}$) on one of the two binomial coefficients to get $n^2 \sum_k \binom{n-1}{k-1}\binom{n-1}{n-k}$, and apply Vandermonde ($-1, n \in \mathbb{Z}$) now. Either way we get $n^2\binom{2n-2}{n-1}$.

4. For $n \in \mathbb{N}_0$, calculate $\sum_{k \geq 0} \frac{1}{k+1}\binom{n}{k}(-1)^{k+1}$.

   This problem is about reindexing, twice. We first reindex the absorption identity to get $\frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{k+1}\binom{n}{k}$. Our sum becomes $\frac{1}{n+1}\sum_{k \geq 0}\binom{n+1}{k+1}(-1)^{k+1}$. We now reindex this sum ($v = k+1$) to get $\frac{1}{n+1}\sum_{v \geq 1}\binom{n+1}{v}(-1)^v$. This is almost exactly the binomial theorem (which applies because $n \in \mathbb{N}_0$); all that's missing is the first term. Hence our sum is $\frac{1}{n+1}\left((-1+1)^{n+1} - 1\right) = \frac{-1}{n+1}$.

5. Calculate $\sum_k (-1)^k k \binom{10+k}{3}\binom{10}{k}$.

   Note that the sum is really for $k \in \mathbb{N}_0$, by considering $\binom{10}{k}$. We first use absorption ($k \in \mathbb{Z}$) to rewrite $k\binom{10}{k} = 10\binom{9}{k-1}$. We use symmetry ($10 + k \in \mathbb{N}_0$) to rewrite $\binom{10+k}{3} = \binom{10+k}{7+k}$. We now use upper negation ($7 + k \in \mathbb{Z}$) to rewrite $\binom{10+k}{7+k} = (-1)^{7+k}\binom{7+k-(10+k)-1}{7+k} = -(-1)^k\binom{-4}{7+k}$. Putting it all together, our sum becomes $-10\sum_k \binom{-4}{7+k}\binom{9}{k-1}$. Finally, we are ready for Vandermonde ($7 + k, k - 1 \in \mathbb{Z}$), which gives $-10\binom{5}{6} = 0$. Whew!